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A geometry for multidimensional integrable systems

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Abstract

A deformed differential calculus is developed based on an associative $\star$-product. In two dimensions the Hamiltonian vector fields model the algebra of pseudo-differential operator, as used in the theory of integrable systems. Thus one obtains a geometric description of the operators. A dual theory is also possible, based on a deformation of differential forms. This calculus is applied to a number of multidimensional integrable systems such as the KP hierarchy, thus obtaining a geometrical description of these systems. The limit in which the deformation disappears corresponds to taking the dispersionless limit in these hierarchies.

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1. Introduction

Is there a common structure behind all integrable systems? There are many different types of integrable systems: $(1+1)$- and $(2+1)$-dimensional evolution equations (such as the KdV and KP equations), chiral and harmonic map equations, integrable dynamical systems (such as the Halphen and Kovalevskaya top equations), integrable non-linear ordinary differential equations (such as the Chazy and Painlevé equations) for example, and all these have various properties associated with their integrability (see, for example, [AC]). However, there is very little in the way of general theory, where the apparently disparate properties of various individual integrable systems could be understood in a consistent and coherent way. Indeed, there is no universal definition of what integrability actually is.

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One idea, proposed by Ward [W], is that such system may all be obtained from the (anti)-self-dual Yang–Mills equations and their generalisation by a process of dimensional reduction. For example, the KdV, NLS, sine-Gordon ad Liouville equations may all be obtained from the (anti)-self-dal Yang–Mills equations with $SL(2,\mathbb{C})$-gauge groups, the only difference being the choice of symmetry group and space–time signature [W,MS]. The key idea is not so much the self-dual Yang–Mills equations themselves, but the existence of a Penrose transform for such fields. Under such a transform the fields 'disappear' into the holomorphic geometry (certain holomorphic vector bundles over regions of an auxiliary complex manifold known as twistor space) [W77]. More prosaically, this provides a geometric framework for the Riemann–Hilbert problems used in the construction of solutions to the duality equations. The existence of this transform has been conjectured to be behind the idea of integrability.

The paradigm (and the original example of such a transform) case from the (anti)-self-dual vacuum equations [Pen]. The following formalism is due to Gindikin [G]. Consider the following system of first-order equations depending on a parameter $\tau=\{\tau_0,\tau_1\}\in\mathbb{C}^2$:

$$\begin{aligned}\omega^1(\tau)&=\omega_0^1\tau_0^k+\cdots+\omega_k^1\tau_1^k,\\ &\ \vdots\\ \omega^{2p}(\tau)&=\omega_0^{2p}\tau_0^k+\cdots+\omega_k^{2p}\tau_1^k,\end{aligned}$$

where ω_j^i are 1-forms. Let $\Omega^k(\tau)$ be the bundle of 2-forms,

$$\Omega^k(\tau)=\omega^1(\tau)\wedge\omega^2(\tau)+\cdots+\omega^{2p-1}(\tau)\wedge\omega^{2p}(\tau),$$

satisfying the conditions:
- the $(p+1)$th exterior power of Ω^k vanishes;
- the pth exterior power of Ω^k is non-degenerate;
- $\mathrm{d}\Omega^k(\tau)=0$.

The bundle of forms then encodes the integrability of the original system. In the special case $k=1,\ p=1$ the metric defined by

$$g=\omega_0^1\omega_1^2-\omega_1^1\omega_0^2$$

has vanishing Ricci tensor and (anti)-self-dual Weyl tensor. The form Ω is related to various structures on the corresponding curved twistor space.

One major problem with this geometric approach to the understanding of integrable systems is to find how systems such as the KP equation

$$(4u_t-12uu_x-u_{xxx})_x=3u_{yy}$$

fit into the scheme. There have been various attempts, some erroneous, to do this, the problem stemming from how to give a geometrical description to the pseudo-differential operators used in the derivation of the KP equation and its hierarchy. However, though there are problems with the KP equation itself, these problems vanish in a particular limit (the dispersionless limit) of the KP equation and one obtains a geometric description of this limiting case. Explicitly, let

$$X = \epsilon x, \qquad Y = \epsilon y, \qquad T = \epsilon t,$$
$$U(X, Y, T) = u(x, y, t),$$

then the KP equation becomes, in the limit $\epsilon \to 0$ the dispersionless (or dKP equation):

$$(4U_T - 12UU_X)_X = 3U_{YY}.$$

This also is integrable, but the description does not use pseudo-differential operators but a more geometrical description in terms of vector fields and differential forms.

The central idea of this paper is the development of a deformed differential calculus based on an associative $\star$-product and its application to the theory of integrable systems. One will obtain an elegant description of the KP hierarchy in terms of vector fields and differential forms rather than the more usual pseudo-differential operator formalism. The advantage of this approach will be twofold: firstly, one retains a geometric description, so, for example, one can go to a dual description in terms of differential forms; secondly in the limit in which the deformation disappears one recovers the dKP equation directly without the need of the somewhat singular limit outlined above.

This deformed calculus with be derived in Section 2 and used in Section 3 where various examples of multidimensional integrable systems (an integrable deformation of the (anti)-self-dual vacuum equations, the KP hierarchy and the Toda hierarchy) will be studied. It will turn out that all these systems may be written in terms of a 2-form Ω satisfying the equations

$$d\Omega = 0, \qquad \Omega \wedge \Omega = 0$$

in analogy to Gindikin's bundle of forms. In Section 4 the geometry of the KP hierarchy will be examined in more detail. This work raises a number of further questions, some of which are outlined in Section 5.

2. Deformed differential geometry

A Poisson manifold $\mathcal{M}$ is endowed with a bilinear skew-symmetric Poisson bracket defined, for $u, v \in C^{\infty}(\mathcal{M})$, by

$$\{u, v\}_{\mathrm{PB}} = \omega^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{1}$$

and with the additional property that

$$\{\{f, g\}_{\mathrm{PB}}, h\}_{\mathrm{PB}} + \{\{g, h\}_{\mathrm{PB}}, f\}_{\mathrm{PB}} + \{\{h, f\}_{\mathrm{PB}}, g\}_{\mathrm{PB}} = 0,$$

this begin known as the Jacobi identity. Further, it will be assumed that $\mathcal{M}$ is a symplectic manifold, that is a Poisson manifold for which the matrix ω^{ij} is of maximal rank. It follows that the dimension of $\mathcal{M}$ must be even, so

$$\dim(\mathcal{M}) = 2N$$

for some integer N. It will be assumed that ω^{ij} is constant and that a basis has been chosen in which

$$\omega^{ij} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

This structure may be used to define a deformation of the above Poisson bracket. For $u, v \in C^\infty(\mathcal{M})$ one defines a new product

$$u \star v = \exp\left(\frac{\kappa}{2}\omega^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}\right) u(\boldsymbol{x})v(\tilde{\boldsymbol{x}})\big|_{x=\tilde{x}}$$

or, on expanding the exponential

$$u \star v = \sum_{s=0}^{\infty} \frac{\kappa^s}{2^s s!}\omega^{i_1 j_1}\cdots\omega^{i_s j_s}\frac{\partial^s u}{\partial x^{i_1}\cdots\partial x^{i_s}}\frac{\partial^s v}{\partial x^{j_1}\cdots\partial x^{j_s}}. \tag{2}$$

With this the deformed, or Moyal bracket, is defined as [Mo]

$$\{u, v\} = \frac{u \star v - v \star u}{\kappa}. \tag{3}$$

Lemma 1. *For constants c and functions $u, v \in C^\infty(\mathcal{M})$:*
(a) $\lim_{\kappa\to 0} u \star v = uv$,
(b) $c \star u = cu$,
(c) $\star$ *is associative,*
(d) $\lim_{\kappa\to 0}\{u, v\} = \{u, v\}_{\mathrm{PB}}$,
(e) $\{u, v\}$ *is bilinear, skew-symmetric and satisfies the Jacobi identity.*

Proof. Straightforward from definitions (1) and (2). Note that the Jacobi identity follows from the associativity of the $\star$-product. □

The original motivation for the introduction of such a bracket came from a description of quantum machanics using phase space variables [Mo]. Here κ is replaced by $-i\hbar$, and $\hbar \to 0$ corresponds to taking the classical limit.

It is necessary to introduce another product. For $u, v \in C^\infty(\mathcal{M})$ define

$$u \circ v = \sum_{s=0}^{\infty} \frac{\kappa^{2s}}{2^{2s}(2s+1)!}\omega^{i_1 j_1}\cdots\omega^{i_{2s} j_{2s}}\frac{\partial^{2s} u}{\partial x^{i_1}\cdots\partial x^{i_{2s}}}\frac{\partial^{2s} v}{\partial x^{j_1}\cdots\partial x^{j}_{2s}}. \tag{4}$$

Lemma 2. *For constants c and function $u, v \in C^\infty(\mathcal{M})$:*
(a) $u \circ v = v \circ u$,
(b) $c \circ u = cu$,
(c) $\lim_{\kappa\to 0} u \circ v = uv$,
(d) $\circ$ *is not associative,*
(e) $2\,\mathrm{d}(\kappa u \circ v)/\mathrm{d}\kappa = u \star v + v \star u$.

Proof. Again, these results follow from definitions (2) and (4). □

With this $\circ$-product a deformed, or quantum, differential calculus will be constructed. A similar calculus has recently been constructed by Fedosov [Fe] using the $\star$-product rather than the $\circ$-product. The reason for the introduction of the new product will become apparent later (the motivation coming from the application of this calculus to multidimensional integrable systems) and rests on the following result.

Proposition 1. *For* $u, v \in C^\infty(\mathcal{M})$,

$$\omega^{rs}\frac{\partial u}{\partial x^r}\circ\frac{\partial v}{\partial x^s} = \{u, v\}.$$

Proof. Follows from definitions (3) and (4):

$$\begin{aligned}
&\omega^{rs}\frac{\partial u}{\partial x^r}\circ\frac{\partial v}{\partial x^s}\\
&=\sum_{s=0}^{\infty}\frac{\kappa^{2s}}{2^{2s}(2s+1)!}\omega^{i_1j_1}\cdots\omega^{i_{2s}j_{2s}}\omega^{rs}\frac{\partial^{2s+1}u}{\partial x^{i_1}\cdots\partial x^{i_{2s}}\partial x^r}\frac{\partial^{2s+1}v}{\partial x^{j_1}\cdots\partial x^{j_{2s}}\partial x^s}\\
&=\frac{2}{\kappa}\sum_{s=0}^{\infty}\frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!}\omega^{i_1j_1}\cdots\omega^{i_{2s+1}j_{2s+1}}\frac{\partial^{2s+1}u}{\partial x^{i_1}\cdots\partial x^{i_{2s+1}}}\frac{\partial^{2s+1}v}{\partial x^{j_1}\cdots\partial x^{j_{2s+1}}}\\
&=\left(\frac{u\star v-v\star u}{\kappa}\right)=\{u, v\}.
\end{aligned}$$

In the limit $\kappa \to 0$ this reduces to definition (1). □

In the simplest case ($N = 1$) these formulae may be easily rewritten using the explicit form of ω^{ij}, coordinates $x^1 = x$, $x^2 = y$:

$$u\star v=\sum_{s=0}^{\infty}\frac{\kappa^s}{2^s s!}\sum_{j=0}^{s}(-1)^j\binom{s}{j}\partial_x^{s-j}\partial_y^j u\partial_x^j\partial_y^{s-j}v, \tag{5}$$

$$u\circ v=\sum_{s=0}^{\infty}\frac{\kappa^{2s}}{2^{2s+1}(2s+1)!}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}\partial_x^{2s-j}\partial_y^j u\partial_x^j\partial_y^{2s-j}v, \tag{6}$$

$$\begin{aligned}
\{u, v\}=&\sum_{s=0}^{\infty}\frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!}\\
&\times\sum_{j=0}^{2s+1}(-1)^j\binom{2s+1}{j}\partial_x^{2s+1-j}\partial_y^j u\partial_x^j\partial^{2s+1-j}v
\end{aligned}\tag{7}$$

for functions $u(x, y), v(x, y) \in C^\infty(\mathcal{M})$.

Example 1. Let $\mathcal{M} = T^2$, the 2-torus. Functions on T^2 may be expanded in terms of basis functions

$$e_{\boldsymbol{m}} = \exp\mathrm{i}(m_1x + m_2y).$$

With these one obtains from (5)–(7):

$$e_{\boldsymbol{m}} \star e_{\boldsymbol{n}} = \exp(\tfrac{1}{2}\kappa(\boldsymbol{m} \times \boldsymbol{n}))e_{\boldsymbol{m}+\boldsymbol{n}},$$
$$\{e_{\boldsymbol{m}}, e_{\boldsymbol{n}}\} = \sinh \tfrac{1}{2}\kappa(\boldsymbol{m} \times \boldsymbol{n})e_{\boldsymbol{m}+\boldsymbol{n}},$$
$$e_{\boldsymbol{m}} \circ e_{\boldsymbol{n}} = 2\frac{\sinh \frac{1}{2}\kappa(\boldsymbol{m} \times \boldsymbol{n})}{\kappa(\boldsymbol{m} \times \boldsymbol{n})}e_{\boldsymbol{m}\times\boldsymbol{n}},$$

where $\boldsymbol{m} \times \boldsymbol{n} = m_2 n_1 - m_1 n_2$.

Given such a symplectic manifold and product one may define tangent and cotagent bundles $T\mathcal{M}$ and $T^*\mathcal{M}$, the inner product between basis elements $(\partial/\partial x^i)$ and $\mathrm{d}x^j$ being given by

$$\left\langle \frac{\partial}{\partial x^i}, \mathrm{d}x^k \right\rangle = \delta_i^j.$$

The first difference is the formula for the inner product between general elements $X \in T_p\mathcal{M}$ and $\omega \in T_p^*\mathcal{M}$,

$$\langle X, \omega \rangle = \left\langle X^i \frac{\partial}{\partial x^i}, \omega_j \, \mathrm{d}x^j \right\rangle, = X^i \circ \omega_j \left\langle \frac{\partial}{\partial x^i}, \mathrm{d}x^k \right\rangle, = X^i \circ \omega_i,$$

i.e. the multiplication being done with the $\circ$-product. Similarly, given a vector field X and functon f one defines

$$Xf = X^i \circ \frac{\partial f}{\partial x^i}, \tag{8}$$

again using the $\circ$-product. The general procedure should already be apparent: the only change to the standard, or undeformed, theory is when objects are combined, this being done with the $\circ$-product. Thus in the $\kappa \to 0$ limit the standard theory is recovered. One may extend this new calculus to general tensor fields. However the extension to an exterior differential calculus is of more interest.

The d-operator, which maps r-forms to $(r+1)$-forms is defined as normal. For example, given a 0-form (i.e. a function) the 1-form $\mathrm{d}f$ is defined by the relation

$$\langle X, \mathrm{d}f \rangle = Xf$$

for all vector fields X. From this follows the formula

$$\mathrm{d}f = \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

The wedge product combines forms and so this will be done using the $\circ$-product, Explictly, if

$$\boldsymbol{A} = A_{i_1 \cdots i_p} \mathrm{d}x_1^i \wedge \cdots \wedge \mathrm{d}x^{i_p},$$
$$\boldsymbol{B} = B_{j_1 \cdots j_q} \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_q},$$

then

$$\boldsymbol{A} \wedge \boldsymbol{B} = A_{[i_1 \cdots i_p} \circ B_{j_1 \cdots j_q]} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_q}.$$

Example 2. Suppose dim $\mathcal{M} = 2$ (i.e. $N = 1$). Then for functions $f(x, y), g(x, y) \in C^\infty(\mathcal{M})$,

$$\mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y, \qquad \mathrm{d}g = g_x\,\mathrm{d}x + g_y\,\mathrm{d}y$$

and hence

$$\mathrm{d}f \wedge \mathrm{d}g = (f_x \circ g_y - f_y \circ g_x)\,\mathrm{d}x \wedge \mathrm{d}y, = \{f, g\}\,\mathrm{d}x \wedge \mathrm{d}y.$$

Note that this uses the symmetry property of the $\circ$-product. Care must be taken in higher dimensions since, as the $\circ$-product is not associative, $\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) \neq (\boldsymbol{A} \wedge \boldsymbol{B}) \wedge \boldsymbol{C}$ for arbitrary forms $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$.

Having defined an exterior differential calculus one may define another intrinsic differential object, namely a Lie derivative $\mathcal{L}_X$ corresponding to some vector field $X \in T\mathcal{M}$. On functions

$$\mathcal{L}_X f = Xf$$

and on vector fields

$$(\mathcal{L}_X Y)^i = X^j \circ \frac{\partial Y^i}{\partial x^j} - Y^j \circ \frac{\partial X^i}{\partial x^j}.$$

This will also be written as $\mathcal{L}_X Y = [X, Y]$, and called the commutator of two vector fields. Using the symmetry of the $\circ$- product it follows that the commutator is antisymmetric. One may extend the definition to more general objects such as tensor fields in such a way that the theory is consistent. For example, for any p-form ω and vector field X,

$$\mathrm{d}(\mathcal{L}_X\omega) = \mathcal{L}_X(\mathrm{d}\omega).$$

Normally one has the relations

$$[X, Y]f - X(Yf) + Y(Xf) = 0,$$
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, Y], X] = 0.$$

The proof of these results uses the associativity of normal (i.e. underformed) multiplication, and so do not hold for the deformed definitions based on the non-associative $\circ$-product. However for an important class of vector fields these results do hold. Given a function $f \in C^\infty(\mathcal{M})$ (the Hamiltonian) the corresponding Hamiltonian vector field X_f is defined by

$$X_f = \omega^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}. \tag{9}$$

Strictly speaking these are local Hamiltonian vector fields, Hamiltonian vector fields having to be defined globally on $\mathcal{M}$.

Lemma 3. *For functions $f, g, h \in C^\infty(\mathcal{M})$ and corresponding Hamiltonian fields X_f, X_g and X_h:*

(a) $X_f h = \{f, h\}$,

(b) $[X_f, X_g] = X_{\{f,g\}}$,
(c) $[X_f, X_g]h = X_f(X_g h) - X_g(X_f h)$,
(d) $[[X_f, X_g], X_h] + [[X_g, X_h], X_f] + [[X_h, X_f], X_g] = 0$.

Proof.
(a) Follows from definitions (8), (9) and Proposition 1.
(b) We have

$$\begin{aligned}
[X_f, X_g]^i &= X_f^j \circ \frac{\partial X_g^i}{\partial x^j} - X_g^j \circ \frac{\partial X_f^i}{\partial x^j} \\
&= \omega^{kj} \frac{\partial f}{\partial x^k} \circ \frac{\partial}{\partial x^j}\left(\omega^{ri} \frac{\partial g}{\partial x^r}\right) \\
&\quad - \omega^{kj} \frac{\partial g}{\partial x^k} \circ \frac{\partial}{\partial x^j}\left(\omega^{ri} \frac{\partial f}{\partial x^r}\right) \\
&= \omega^{ri} \frac{\partial}{\partial x^r}\left(\omega^{kj} \frac{\partial f}{\partial x^k} \circ \frac{\partial g}{\partial x^j}\right) = X^i_{\{f,g\}}.
\end{aligned}$$

This uses the antisymmetry of ω^{ij} and the symmetry of the $\circ$-product.
(c) We have

$$\begin{aligned}
[X_f, X_g]h &= X_{\{f,g\}}h = \{\{f, g\}h\} \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} = X_f(X_g h) - X_g(X_f h).
\end{aligned}$$

This uses result (b), the Jacobi identity for the Moyal bracket and the antisymmetry of the Moyal bracket (Lemma 1).
(d) Follows from the Jacobi identity for the Moyal bracket (Lemma 1). □

Example 3. For two-dimensional manifolds $\mathcal{M}^2$ these Hamiltonian vector fields generate the Lie algebra of area preserving diffeomorphisms of the manifold where the area element is $\mathrm{d}x \wedge \mathrm{d}y$ and the composition of two Hamiltonian vector fields is defined to be the Lie bracket of these fields. Explicitly, the field X_f generates the infinitesimal transformation $x \to x - \epsilon f_y$, $y \to y + \epsilon f_x$. This Lie algebra will be denoted by $\mathrm{sdiff}_\kappa(\mathcal{M}^2)$.

The differential objects constructed have been intrinsic to the manifold. One should be able to introduce a connection on $\mathcal{M}$ and define a covariant differentiation and hence curvature. A similar programme has been carried out using the $\star$-product by Vasiliev [V]. However for the application of this calculus to the theory of integrable systems such a structure will not be required.

This $\star$-product is essentially unique. For a product

$$f \star g = fg + \sum_{r=1}^{\infty} \kappa^r Q_r(f, g)$$

(where the Q_r are bilinear differential operators) to be associative places strong restrictions on the type of higher-order terms that may be added. Further, the requirement that the bracket defined by $\{f, g\}' = (f \star g - g \star f)/\kappa$ should reduce to the Poisson bracket in the $\kappa \to 0$ limit introduces further restrictions and from these considerations follow various results on the uniqueness of the Moyal bracket [A,BFFLS,F1]. However these uniqueness results only state that any such deformations are equivalent to the Moyal bracket; there are apparently different structures which, after various changes of variable, become the Moyal bracket. For example one may define the following associative $\star$-product (in the $N = 1$ case):

$$f \star g = \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \frac{\partial^s f}{\partial x^s} \frac{\partial^s g}{\partial y^s}. \tag{10}$$

This also defines a bracket

$$\{f, g\}' = \frac{f \star g - g \star f}{\kappa}, \tag{11}$$

which reduces to the standard Poisson bracket in the $\kappa \to 0$ limit. This new bracket will be called the Kupershmidt–Manin bracket [K,Ma]. As above, one may define a corresponding $\circ$-product

$$f \circ g = \sum_{s=0}^{\infty} \frac{\kappa^s}{(s+1)!} \sum_{m=0}^{s} \partial_x^{s-m} \partial_y^m f \partial_y^{s-m} \partial_x^m g, \tag{12}$$

so that

$$\omega^{rs} \frac{\partial u}{\partial x^r} \circ \frac{\partial v}{\partial x^s} = \{f, g\}'$$

and hence an equivalent deformed differential geometry based on these new structures. The form of the $\star$-product is somewhat simplier then that given by (5), though the dependence on the sympletic structure of $\mathcal{M}$ is less transparent. The importance of these new products comes from their relationship to the algebra of pseudo-differential operators. A pseudo-differential operator P is an operator of the form

$$P = \sum_{j=-\infty}^{\text{finite}} a_j(x) \partial^j,$$

where the multiplication of two such operators uses the generalised Leibnitz rule

$$\partial^m a = a \partial^m + \sum_{k=0}^{\infty} \frac{m(m-1)\cdots(m-k-1)}{k!} \partial^k a \partial^{m-k}.$$

The set of such operators will be denoted by $\mathcal{P}$. The symbol of a pseudo-differential opeator is a function of two variable defined by

$$\operatorname{sym}\left(\sum_{j=-\infty}^{\text{finite}} a_j(x) \partial^j \right) = \sum_{j=-\infty}^{\text{finite}} a_j(x) y^j.$$

It has the important property that for all $P, Q \in \mathcal{P}$

$$\operatorname{sym}(PQ) = \operatorname{sym}(P) \star_{\kappa=1} \operatorname{sym}(Q),$$

where $\star_{\kappa=1}$ denotes the $\star$-product (10) evaluated at $\kappa = 1$. It follows from this that

$$\mathrm{sym}([P, Q]) = \{\mathrm{sym}(P), \mathrm{sym}(Q)\}'_{\kappa=1},$$

where $[P, Q] = PQ - QP$. Thus one may replace pseudo-differential operators and its corresponding algebra by functions of two-variables where the composition of two functions is done with the Kupershmidt–Manin bracket evaluated at $\kappa = 1$. More details of these algebraic properties may be found in [F-FMR]. Using the ideas developed above one may give these pseudo-differential operators a geometrical interpretation.

Theorem 1. *Let $\mathcal{H}$ be the space of Hamiltonian vector fields (where $N = 1$) whose Hamiltonians have Laurent expansions in the variable y,*

$$\mathcal{H} = \left\{ X_f \colon f = \sum_{j=-\infty}^{\text{finite}} a_j(x) y^j \right\}.$$

Then the map

$$\iota : \mathcal{P}/\{\text{constants}\} \to \mathcal{H}$$

given by

$$\iota(P) = X_{\mathrm{sym}}(P)$$

is an isomorphism. Moreover

$$\iota([P, Q]) = [X_{\mathrm{sym}(P)}, X_{\mathrm{sym}(Q)}] = X_{\{\mathrm{sym}(P), \mathrm{sym}(Q)\}'_{\kappa=1}},$$

where the Lie bracket of vector fields is evaluated using the product $\circ_{\kappa=1}$ given by (12) evaluated at $\kappa = 1$.

Proof. Straightforward. Given a Hamiltonian vector field one can construct the corresponding Hamiltonian (up to a constant) and hence a pseudo-differential operator whose symbol is the Hamiltonian. Conversely, $\iota(P + c) = \iota(P)$ for all $P \in \mathcal{P}$. The last part of the theorem follows from the properties of the symbol map. □

The set of Hamiltonian vector fields clearly forms a Lie algebra under the composition defined by the Lie bracket. One may define the adjoint representation as follows. For functions $f, g, F \in C^\infty(\mathcal{M})$ define

$$\mathrm{ad}(f)g = \{f, g\}, \qquad \mathrm{Ad}(F)g = F \star g \star F^{-1},$$

the connection between the two coming from the deformed exponential

$$\exp_\kappa f = 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f \star \cdots \star f}_{n-\text{terms}}.$$

So, if $F = \exp_\kappa f$,

$$\mathrm{Ad}(F)g = \exp_\kappa(\mathrm{ad}(f))g$$

(this uses the Baker–Campbell–Hausdorff formula). On vector fields,

$$\mathrm{ad}(X_f)X_g = X_{\{f,g\}}, \qquad \mathrm{Ad}(F)X_g = X_{F\star g\star F^{-1}}.$$

These will be used in Section 4 to describe the dressing properties of the KP hierarchy.

The residue of a pseudo-differential operator $P = \sum a_n \partial^n$ is defined by

$$\mathrm{res}(P) = a_{-1}.$$

It follows that

$$\mathrm{res}(P) = \mathrm{res}(\mathrm{sym}(P)) = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{sym}(P)\, \mathrm{d}y,$$

where the residue of the function $\mathrm{sym}(P)$ is the normal residue, regarding $\mathrm{sym}(P)$ as a complex function of y. The residue has many uses, in particular in the study of the Hamiltonian properties of integrable systems.

3. Applications to integrable systems

In this section a number of multidimensional integrable systems will be studied using the geometric structures developed in Section 2. It will be shown that these systems may all be written in terms of a 2-form Ω which satisfies equations

$$\mathrm{d}\Omega = 0, \qquad \Omega \wedge \Omega = 0.$$

These equations encode the integrability conditions for these systems in an elegant geometric manner.

Let $\mathcal{M}$ be a sympletic manifold with some associated $\star$-product. In applications one will require an extended manifold

$$\widetilde{M} = \mathcal{M} \oplus \mathcal{T}$$

where $\mathcal{T}$ consists of an extra set of coordinates (for example the 'times' in a hierarchy of evolution equations). The manifold $\mathcal{M}$ may be thought of as a phase space and in the applications considered here this will be two-dimensional. The $\star$-product extends to a product on $\widetilde{\mathcal{M}}$ by

$$u(\boldsymbol{x}, \boldsymbol{t}) \star v(\boldsymbol{x}, \boldsymbol{t}) = \exp\left(\kappa \omega^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}\right) u(\boldsymbol{x}, \boldsymbol{t}) v(\tilde{\boldsymbol{x}}, \boldsymbol{t})\big|_{\boldsymbol{x}=\tilde{\boldsymbol{x}}}$$

(where $\boldsymbol{x} = \{x^i\}$ are coordinates on $\mathcal{M}$ and $\boldsymbol{t}$ are coordinates on $\mathcal{T}$), that is, the dependence on the coordinates on $\mathcal{T}$ is ignored. The differential calculus outlined in Section 2 similarly extends to the manifold $\widetilde{\mathcal{M}}$. One difference is that X_f will refer to a Hamiltonian vector field on $\mathcal{M}$ whose Hamiltonian function depends on the coordinates on $\widetilde{\mathcal{M}}$,

$$X_f = \omega^{ij} \frac{\partial f(\boldsymbol{x}, \boldsymbol{t})}{\partial x^i} \frac{\partial}{\partial x^j},$$

i.e. a time dependent Hamiltonian vector field on $\mathcal{M}$, where the 'times' are the coordinates on $\mathcal{T}$.

3.1. The anti-self-dual vacuum equations

The anti-self-dual vacuum equations govern the behaviour of complex 4-metrics of signature $(+,+,+,+)$ whose Ricci tensor is zero and whose Weyl tensor is anti-self-dual. Since these curvature conditions are invariant under changes of coordinates there are many ways to write these equations. One particular form of the equations uses the face that such metrics are automatically Kähler and so may be written in terms of a single scalar function Ω, the Kähler potential. The curvature conditions then give the equation governing the potential (known as Plebanski's 1st Heavenly equation [P1]):

$$\frac{\partial^2\Omega}{\partial x\partial\tilde{x}}\frac{\partial^2}{\partial y\partial\tilde{y}}-\frac{\partial^2\Omega}{\partial x\partial\tilde{y}}\frac{\partial^2\Omega}{\partial y\partial\tilde{x}}=1. \tag{13}$$

The corresponding anti-self-dual Ricci-flat metric is

$$g(\Omega)=\frac{\partial^2\Omega}{\partial x^i\partial\tilde{x}^j}\,\mathrm{d}x^i\,\mathrm{d}\tilde{x}^j,\quad \tilde{x}^i=\tilde{x},\tilde{y},\quad x^j=x,y. \tag{14}$$

This equation can, in principle, be solved using a Penrose transform – the orignal non-linear graviton construction [Pen]. Although not realised at the time, the existence of such a transform makes (13) into a completely integrable system, an important, and rare, example of a multidimensional integrable system. As such it has all the properties one would expect of such a system, an infinite number of conservation laws [S93] and an associated hierarchy [S95b], for example. A Lax pair for this equation was derived by Newman et al. [NPT] and later interpreted by Park [Pa] as that for a two-dimensional topological chiral model with gauge potentials in the infinite-dimensional Lie algebra $s\,\mathrm{diff}(\mathcal{M}^2)$ for some two-dimensional manifold $\mathcal{M}$.

The equation may be written as

$$\{\Omega_x,\Omega_y\}_{\mathrm{PB}}=1,$$

where the Poisson bracket is defined by

$$\{f,g\}_{\mathrm{PB}}=f_{\tilde{x}}g_{\tilde{y}}-f_{\tilde{y}}g_{\tilde{x}}.$$

The equation that will be studied in this section is an integrable deformation of this equation, where the Poisson bracket has been replaced by the Moyal bracket (3)

$$\{\Omega_x,\Omega_y\}=1. \tag{15}$$

The space $\mathcal{M}$ will have coordinates $\{\tilde{x},\tilde{y}\}$ (with composition using the products (2) and (4)) and the space $\mathcal{T}$ will be taken to be $\mathbb{R}^2$ (or possibly $\mathbb{C}^2$) with coordinates $\{x,y\}$. This deformed system (15) will be studied using the deformed calculus developed in Section 2. It will turn out that it shares many of the features and properties of the underformed system (13).

Let $\mathcal{U}$ and $\mathcal{V}$ be the following vector fields on $T\widetilde{\mathcal{M}}$:

$$\mathcal{U}=\lambda\frac{\partial}{\partial x}+X_f,\qquad \mathcal{V}=\lambda\frac{\partial}{\partial y}+X_g,$$

where $\lambda \in \mathbb{CP}^1$ is a constant known as the spectral parameter. The system of equations for the function $\psi \in C^\infty(\widetilde{\mathcal{M}})$,

$$\mathcal{U}(\psi) = 0, \qquad \mathcal{V}(\psi) = 0$$

(or, equivalently,

$$\lambda\psi_x + \{f, \psi\} = 0, \qquad \lambda\psi_y + \{g, \psi\} = 0),$$

is overdetermined unless the integrability condition

$$[\mathcal{U}, \mathcal{V}] = 0$$

holds. Here [,] is the Lie bracket of vector fields. If these equations are satisfied, then one has two independent solutions L and M for ψ which satisfy the equation $\{L, M\} = 1$. Note that the above equations may be written in the following ways:

$$\mathcal{U}(L) = \mathcal{U}(M) = 0, \qquad \mathcal{V}(L) = \mathcal{V}(M) = 0,$$

or

$$\langle \mathcal{U}, \mathrm{d}L\rangle = \langle \mathcal{U}, \mathrm{d}M\rangle = 0, \qquad \langle \mathcal{V}, \mathrm{d}L\rangle = \langle \mathcal{V}, \mathrm{d}M\rangle = 0.$$

The above integrability conditions are satisfied if the functons f and g satisfy the equations

$$f_y - g_x = 0, \qquad \{f, g\} = 1.$$

The first equation implies the existence of a scalar function Ω such that $f = \Omega_x$, $g = \Omega_y$ and with these the second equation becomes the deformed Plebanski equation (15). This shows that this may be interpreted as a two-dimensional chiral model with gauge potentials in the Lie algebra $s\ \mathrm{diff}_\kappa(\mathcal{M}^2)$.

In [S92] the vector fields $\mathcal{U}$ and $\mathcal{V}$ were interpretated as operators:

$$\begin{aligned}\mathcal{U} = \lambda\partial_x + \sum_{s=0}^{\infty} \frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!} \\ \times \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} \partial_{\tilde{x}}^{2s+1-j} \partial_{\tilde{y}}^{j} f \partial_{\tilde{x}}^{j} \partial_{\tilde{y}}^{2s+1-j}, \\ \mathcal{V} = \lambda\partial_y + \sum_{s=0}^{\infty} \frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!} \\ \times \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} \partial_{\tilde{x}}^{2s+1-j} \partial_{\tilde{y}}^{j} g \partial_{\tilde{x}}^{j} \partial_{\tilde{y}}^{2s+1-j}.\end{aligned}$$

The geometrical approach used here is much simplier, and the manipulations using the $\circ$-product which lead to Eq. (15) are almost transparent, deviating very little from the underformed calculation which leads to Eq. (13) (to achieve such a result was one of the original motivations in the development of the deformed calculus). Another advantage of the geometrical over the operator based approach is that one can go over to a dual description in terms of differential forms on $T^*(\widetilde{\mathcal{M}})$.

Let Ω be the 2-form

$$\Omega = \mathrm{d}x \wedge \mathrm{d}y + \lambda(\Omega_{x\tilde{x}}\,\mathrm{d}x \wedge \mathrm{d}\tilde{x} + \Omega_{x\tilde{y}}\,\mathrm{d}x \wedge \mathrm{d}\tilde{y} + \Omega_{y\tilde{x}}\,\mathrm{d}y \wedge \mathrm{d}\tilde{x} + \Omega_{y\tilde{y}}\,\mathrm{d}y \wedge \mathrm{d}\tilde{y}) + \lambda^2\,\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}.$$

This clearly satisfies the condition $\mathrm{d}\Omega = 0$, and in addition

$$\begin{aligned}\Omega \wedge \Omega &= \lambda^2(\Omega_{x\tilde{x}} \circ \Omega_{yy} - \Omega_{x\tilde{y}} \circ \Omega_{y\tilde{x}} - 1)\,\mathrm{d}x \wedge \mathrm{d}\tilde{x} \wedge \mathrm{d}y \wedge \mathrm{d}\tilde{y}\\ &= \lambda^2(\{\Omega_x, \Omega_y\} - 1)\,\mathrm{d}x \wedge \mathrm{d}\tilde{x} \wedge \mathrm{d}y \wedge \mathrm{d}\tilde{y}\\ &= 0\end{aligned}$$

by virtue of (15). Thus the Lax pair, and hence the integrability of this deformed system is encoded into the equations

$$\mathrm{d}\Omega = 0, \qquad \Omega \wedge \Omega = 0.$$

Further properties of this system have been found. In [S92] a perturbative solution in powers of κ was constructed. On writing

$$\Omega = \sum_{n=0}^{\infty} \kappa^n \Omega_n,$$

one obtains Plebasnki's equation (13) for Ω_0 and an infinite number of linear equations for the Ω_n, $n > 0$, of the form

$$\Box_{\Omega_0}\Omega_n = S_n[\Omega_0, \ldots, \Omega_{n-1}], \quad n = 1, 2, \ldots, \infty.$$

The operator $\Box_\Omega$ is the wave operator on the space–time given by the metric $\boldsymbol{g}(\Omega)$ given by Eq. (14) and the function S_n is some known function of its arguments. A similar procedure may be applied to a Moyal-algebraic deformation of Plebanski's 2nd heavenly equation [PPRT]. In [C] a symmetry reduction of this system was studied and in [T94a] the dressing transform (using a Riemann-Hilbert factorisation in the corresponding Moyal loop group) was constructed. As mentioned earlier, one may study the conservation laws, symmetries and hierarchies associated with Plebanski's equation and these results still holds, under the replacement of the Poisson bracket by the Moyal bracket, for the deformed system (15).

A slightly more general framework may be achieved by observing that the vector fields ∂_x, ∂_y, X_f and X_g which make up the vector fields $\mathcal{U}$ and $\mathcal{V}$ all preserve the volume form $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$ on $\mathcal{M}$, as in the construction of self-dual metrics [MN].

3.2. KP hierarchy

The KP hierarchy is defined as follows. Let $\mathcal{L}$ be the pseudo-differential operator

$$\mathcal{L} = \partial + \sum_{n=1}^{\infty} u_n(x, \boldsymbol{t})\partial^{-n},$$

where $\boldsymbol{t} = \{t_1, t_2 \ldots\}$. The evolution of the fields $u_n(x, \boldsymbol{t})$ with respect to the times $\boldsymbol{t}$ is given by the Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = [\mathcal{B}_n, \mathcal{L}], \tag{16}$$

where

$$\mathcal{B}_n = [\mathcal{L}^n]_+, \quad n = 1, 2 \ldots, \infty$$

and $[\mathcal{O}]_+$ denotes the projection onto the purely differential part of the pseudo-differential operator $\mathcal{O}$. Similarly, $[\mathcal{O}]_-$ denotes the projection onto purely negative powers, so $\mathcal{O} = \mathcal{O}_+ + \mathcal{O}_-$.

Let the coordinates on $\mathcal{M}$ be $\{x, \lambda\}$ and times $\boldsymbol{t}$ be coordinates on $\mathcal{T}$. The $\star$- and $\circ$-products on $\mathcal{M}$ will be given by Eqs. (10) and (12) evaluated at $\kappa = 1$, and the $\prime$ will be dropped on the corresponding bracket for notional convenience. Taking the symbols of the above operators gives the following functions on $\widetilde{\mathcal{M}}$:

$$L = \mathrm{sym}(\mathcal{L}) = \lambda + \sum_{n=1}^{\infty} u_n(x, \boldsymbol{t})\lambda^{-n}, \qquad B_n = \mathrm{sym}(\mathcal{B}_n).$$

Note that

$$\mathrm{sym}(\mathcal{L}^n) = \mathrm{sym}(\mathcal{L}) \star_{\kappa=1} \cdots \star_{\kappa=1} \mathrm{sym}(\mathcal{L})$$

(which does not equal $\mathrm{sym}(\mathcal{L})^n$) which Kupershmidt [K] denotes by $L^{\star n}$ (see also [F-FMR]). The Lax equation (16) becomes the vector field equation

$$L_n(L) = 0, \quad n = 1, 2, \ldots, \infty. \tag{17}$$

Here $L_n \in T\widetilde{\mathcal{M}}$ is the vector field

$$L_n = \frac{\partial}{\partial t_n} - X_{B_n}$$

and the operations are performed with $\kappa = 1$. At this point this condition will be dropped, thus obtaining a κ-dependent KP hierarchy. The Lax function L remains unchanged, but the B_n acquire κ dependence, since they are now defined by the formula

$$B_n = [L^{\star n}]_+$$

(and so reduce to the previous definition if $\kappa = 1$), where $_+$ denotes the projection onto non-negative powers of λ. This has the advantage that one may recover the dispersionless KP hierarchy in the $\kappa \to 0$ limit without the need for rescaling variables.

Example 4. The first few equations in the κ-dependent KP hierarchy (17) are [K90]:

$$\begin{aligned} B_1 &= \lambda, \\ B_2 &= \lambda^2 + 2u_2, \\ B_3 &= \lambda^3 + 3\lambda u_1 + 3u_2 + 3\kappa u_{1,x}, \end{aligned}$$

which leads to the evolution equations

$$\begin{aligned} u_{1,t_2} &= 2u_{2,x} + \kappa u_{1,xx}, \\ u_{2,t_2} &= 2u_{3,x} + 2u_1 u_{1,x} + \kappa u_{2,xx}, \\ u_{1,t_3} &= 3u_{3,x} + 6u_1 u_1, x + 3\kappa u_{2,xx} + \kappa^2 u_{1,xxx} \end{aligned}$$

(the t_1-flows are trivial). These show the κ-dependent terms. On eliminating u_2 and u_3 one obtains the KP equation

$$(4u_{1,t_3} - 12u_1 u_{1,x} - \kappa^2 u_{1,xxx})_x = 3u_{1,t_2t_2}.$$

One may obtain an equivalent KP hierarchy by using the Moyal $\star$-product and bracket rather than the Kupershmidt–Manin $\star$-product and bracket [K,S95a]. One obtains the functions

$$\begin{aligned} B_1 &= \lambda, \\ B_2 &= \lambda^2 + 2u_1, \\ B_3 &= \lambda^3 + 3\lambda u_1 + 3u_2 \end{aligned}$$

(κ-dependent terms only appear in the B_n for $n > 3$) and evolution equations

$$\begin{aligned} u_{1,t_2} &= 2u_{2,x}, \\ u_{2,t_2} &= 2u_1 u_{1,x} + 2u_{3,x}, \\ u_{1,t_3} &= 6u_1 u_{1,x} + 3u_{3,x} + \tfrac{1}{4}\kappa^2 u_{1,xxx}. \end{aligned}$$

This system also leads to the KP equation on eliminating u_2 and u_3, and on redefining the fields it is easy to see that these two systems are equivalent. Note that in both cases the limit $\kappa \to 0$ one obtains the dispersionless KP equation directly without further rescaling of the variables.

An alternative form of the KP hierarchy is based on the zero-curvature conditions (which follow from the Lax equation (16))

$$\frac{\partial \mathcal{B}_n}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial t_n} + [\mathcal{B}_n, \mathcal{B}_m] = 0,$$

or equivalently

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + \{B_n, B_m\} = 0. \tag{18}$$

Note that this is the condition of the vector fields L_n to commute, $[L_m, L_n] = 0$ for all $m, n = 1, 2, \ldots, \infty$.

These zero-curvature relations may be encoded in a 2-form Ω defined by

$$\Omega = \mathrm{d}\lambda \wedge \mathrm{d}x + \sum_{n=2}^{\infty} \mathrm{d}B_n \wedge \mathrm{d}t_n.$$

This form satisfies the following equations:

$$\mathrm{d}\Omega = 0, \qquad \Omega \wedge \Omega = 0.$$

The first is obvious. One has to be careful in evaluating the second equation (see Example 2), but one obtains

$$\Omega \wedge \Omega = \sum_{m,n=2}^{\infty} \left[\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + \{B_n, B_m\} \right] \mathrm{d}\lambda \wedge \mathrm{d}x \wedge \mathrm{d}t_m \wedge \mathrm{d}t_n$$

and hence $\Omega \wedge \Omega = 0$ if and only if the zero-curvature relations (18) hold. The geometry of the KP hierarchy will be discussed further in Section 4.

3.3. *Toda hierarchy*

The definition of this hierarchy is very similar to the definition of the KP hierarchy. The Lax operator is

$$\mathcal{L} = \mathrm{e}^{\partial} + \sum_{n=0}^{\infty} u_n(x,t)\mathrm{e}^{-n\partial}$$

(note the range of summation) and the evolutions of the fields are given by the Lax equation

$$\frac{\partial \mathcal{L}}{\partial t_n} = [\mathcal{B}_n, \mathcal{L}],$$

where

$$\mathcal{B}_n = [\mathcal{L}^n]_+, \quad n = 1, 2, \ldots, \infty$$

and $[\mathcal{O}]_+$ denotes the projection onto positive powers of e^{∂} of the pseudo-differential operator $\mathcal{O}$. The operator e^{∂} acts as a shift operator,

$$\mathrm{e}^{n\partial} f(x) = f(x+n).$$

The geometric description of the hierarchy is obtained in the same way as above. Taking symbols of the operators give

$$L = \mathrm{sym}(\mathcal{L}), = \mathrm{e}^{\lambda} + \sum_{n=0}^{\infty} \mathrm{e}^{-n\lambda},$$

$$B_n = \mathrm{sym}(\mathcal{B}_n)$$

and the above Lax equation becomes

$$\frac{\partial L}{\partial t_n} = \{B_n, L\}. \tag{19}$$

Once again the condition $\kappa = 1$ will be dropped, so composition will be done using the Kupershmidt–Manin $\star$- and $\circ$-products (10) and (12), so now the B_n are defined by the equation $B_n = [L^{\star n}]_+$ where $_+$ dentoes the projection onto non-negative powers of e^{λ}, as in the κ-dependent KP hierarchy. In the limit $\kappa \to 0$ one obtains the dispersionless Toda hierarchy. One difference between the hierarchy and the KP hierarchy is that the evolution equations for the fields contains an infinite number of κ-dependent terms. However these may be recombined in terms of shift operators, as the following example will show.

Example 5. One possible truncation of this hierarchy is to set $u_n = 0$ for $n \geq 2$, so that

$$L = e^{\lambda} + u_0 + u_1 e^{-\lambda}, \qquad B_1 = e^{\lambda} + u_0.$$

The evolution equations for the fields u_0 and u_1 are given by

$$\frac{\partial L}{\partial t} = \{B_1, L\},$$

where the bracket is the Kupershmidt–Manin bracket (11) and, for greater generality, the $\kappa = 1$ condition has been dropped.

This gives the equations

$$u_{0,t}(x) = \frac{1}{\kappa}\left[\sum_{s=0}^{\infty} \frac{\kappa^s}{s!}\partial_x^s - 1\right] u_1(x) = \frac{u_1(x+\kappa) - u_1(x)}{\kappa},$$

$$u_{1,t}(x) = \frac{u_1(x)}{\kappa}\left[1 - \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^s}{s!}\partial_x^s\right] u_0(x) = \frac{u_1(x)[u_0(x) - u_0(x-\kappa)]}{\kappa}$$

and on eliminating u_0 one obtains the Toda lattice equation

$$(\log u_1(x))_{tt} = \frac{u_1(x+\kappa) - 2u_1(x) + u_1(x-\kappa)}{\kappa^2}.$$

Instead of using the Kupershmidt–Manin bracket one could use the Moyal bracket. This gives the slightly different equations

$$u_{0,t}(x) = \frac{2}{\kappa}\left[\sum_{s=0}^{\infty} \frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!}\partial_x^{2s+1}\right] u_1(x)$$

$$= \frac{1}{\kappa}\left[u_1(x + \tfrac{1}{2}\kappa) - u_1(x - \tfrac{1}{2}\kappa)\right],$$

$$u_{1,t}(x) = \frac{2u_1(x)}{\kappa}\left[\sum_{s=0}^{\infty} \frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!}\partial_x^{2s+1}\right] u_0(x)$$

$$= \frac{u_1}{\kappa}\left[u_0(x + \tfrac{1}{2}\kappa) - u_0(x - \tfrac{1}{2}\kappa)\right].$$

On eliminating u_0 one again recovers the Toda lattice equation. Note that with the Kupershmidt–Manin bracket one obtains a forward/backward difference operator while the Moyal bracket gives a central difference operator. In both cases κ acts as the lattice spacing and as $\kappa \to 0$ one recovers the dispersionless Toda equations (since both these brackets are deformations of the Poisson bracket)

$$u_{0,t} = u_{1,x}, \qquad u_{1,t} = u_1 u_{0,x}.$$

Some of the properties of this system and its hierarchy may be found in [K85,FS].

As with the KP hierarchy, the Lax equation (19) is equivalent to a set of zero-curvature relations for the B_n and these may be encoded into a 2-form Ω which satisfies the equations

$$\mathrm{d}\Omega = 0, \qquad \Omega \wedge \Omega = 0$$

in exactly the same way as was done for the KP hierarchy.

4. Geometry of the KP hierarchy

The main result of the section is to show how a solution of the KP hierarchy may be associated to a solution of a Riemann–Hilbert problem in the Lie group $S\ \mathrm{Diff}_\kappa(\mathcal{M}^2)$ (the Lie group corresponding to the Lie algebra $s\ \mathrm{diff}_\kappa(\mathcal{M}^2)$. Explicitly, given a map

$$\begin{pmatrix} x \\ k \end{pmatrix} \mapsto \begin{pmatrix} f(x,k) \\ g(x,k) \end{pmatrix}$$

with $\{f, g\} = 1$ then this map factors, so there exists a map

$$\begin{pmatrix} P \\ Q \end{pmatrix} \mapsto \begin{pmatrix} f(P,Q) \\ g(P,Q) \end{pmatrix},$$

where the right-hand side is analytic in k (the notation S_- will be used to denote the part of the Laurent series S consisting of negative powers of k only). The results derived in Section 2 enable existing results on the KP hierarchy to be lifted whilst furnishing them with a geometrical interpretation (the definitions of the manifold $\widetilde{\mathcal{M}}$ and $\star$- and $\circ$-products will be the same as in Section 3.2). In this section $\kappa = 1$ and the κ-symbol on the exponential $\exp_\kappa$ will be dropped. The main results of this section are due to Takasaki and Takebe [TT]. A more careful analysis is needed for $\kappa \neq 1$.

More fundamental than the Lax operator $\mathcal{L}$ is the operator $\mathcal{W}$ defined by

$$\mathcal{W} = 1 + \sum_{n=1}^{\infty} w_n \partial^{-n}$$

with which the Lax operator is defined as

$$\mathcal{L} = \mathcal{W}\partial\mathcal{W}^{-1}.$$

The evolution of the fields w_n are given by the equation

$$\frac{\partial \mathcal{W}}{\partial t_n} = \mathcal{B}_n \mathcal{W} - \mathcal{W}\partial^n$$

from which follows the Lax equation (16). On taking the symbols of the operators one obtains

$$W = \mathrm{sym}(\mathcal{W}) = 1 + \sum_{n=1}^{\infty} w_n k^{-n},$$
$$L = \mathrm{sym}(\mathcal{L}) = \mathrm{Ad}(W)k.$$

The Orlov operator $\mathcal{M}$ is defined by [GO]

$$\mathcal{M} = \mathcal{W}\left(\sum_{n=1}^{\infty} n t_n \partial^{-n} + x\right)\mathcal{W}^{-1}$$

or, equivalently, by

$$M = \mathrm{sym}(\mathcal{M}) = \mathrm{Ad}(W \exp[\boldsymbol{t}(k)])x,$$

where $\boldsymbol{t}(k) = \sum_{n=1}^{\infty} t_n k^n$.

Lemma 4. *The pair (L, M) satisfy the equations*

$$\frac{\partial L}{\partial t_n} = \{B_n, L\}, \qquad \frac{\partial M}{\partial t_n} = \{B_n, M\}, \qquad \{L, M\} = 1.$$

Conversely, given such a pair then there exists a unique dressing function W so that $L = \mathrm{Ad}(W)k$ and $M = \mathrm{Ad}(W \exp[\boldsymbol{t}(k)])x$.

Such a (L, M) pair will be said to satisfy the KP hierarchy. The first two of these equations may be re-written as vector field equations:

$$(\partial_{t_n} - X_{B_n})L = 0, \qquad (\partial_{t_n} - X_{B_n})M = 0,$$

or alternatively, using the inner product $\langle\ ,\ \rangle$ between vector fields and forms, as

$$\langle \partial_{t_n} - X_{B_n}, \mathrm{d}L\rangle = 0, \qquad \langle \partial_{t_n} - X_{B_n}, \mathrm{d}M\rangle = 0$$

for $n = 1, 2, \ldots, \infty$. These functions (L, M) will play the analogous roles to the coordinates on the twistor surfaces in the non-linear graviton construction.

The next theorems show how such a pair is related to a Riemann–Hilbert factorisation problem. The first one shows how a solution to the Riemann–Hilbert problem defines a solution to the KP hierarchy and the second one shows the converse.

Theorem 2. *Suppose one has functions*

$$L = \mathrm{Ad}(W)k, \qquad M = \mathrm{Ad}(W \exp[\boldsymbol{t}(k)])x$$

(with $\{L, M\} = 1$). Then for any pair of functions $f(x, k)$, $g(x, k)$ (with Laurent series if

$$\{f, g\} = 1, \qquad f(M, L)_- = 0, \qquad g(M, L)_- = 0,$$

then the pair (L, M) satisfies the KP hierarchy.

Theorem 3. *If the pair (L, M) satisfies the KP hierarchy then there exist functions f, g such that*

$$\{f, g\} = 1, \qquad f(M, L)_- = 0, \qquad g(M, L)_- = 0.$$

The proofs are basically identical to the proofs in [TT], the only difference being that here they are reformulated in terms of the deformed geometric structures rather than in terms of pseudo-differential operators, One may also prove the uniqueness results for the solution (L, M), at least in the neighbourhood of the trivial solution (k, x).

One outstanding problem is how to relate the 2-form Ω to the pair (L, M). In the dispersionless limit one has (for all the systems discussed in Section 3) a relation

$$\Omega = \mathrm{d}L \wedge \mathrm{d}M.$$

However, the proof of this result relies on the associative property of normal multiplication which no longer holds for the deformed $\circ$-product. It may be that this result still holds, for example, the obstruction might vanish due to the relation $\{L, M\} = 1$. This problem, of how to understand the direct relation between the pair (L, M) and Ω is currently under investigation.

5. Comments

In summary, the three classes of integrable hierarchy discussed in Section 3 may all be formulated in terms of vector fields $\mathcal{V}_i$ which preserve a volume form

$$\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \bigwedge_{n=1}^{\infty} \mathrm{d}t_n$$

(or $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$ in the deformed anti-self-dual vacuum equations) on $\mathcal{M}$, together with function (L, M) which satisfy the equations

$$\mathcal{V}_i(L) = 0, \qquad \mathcal{V}_i(M) = 0,$$

or, in terms of the inner product between $T\mathcal{M}$ and $T^*\mathcal{M}$,

$$\langle \mathcal{V}_i, \mathrm{d}L \rangle = 0, \qquad \langle \mathcal{V}_i, \mathrm{d}M \rangle = 0$$

with the functions L and M being independent: $\{L, M\} = 1$. A dual description also exists for all these systems in terms of a 2-form Ω on $\mathcal{M}$ satisfying the equations

$$\mathrm{d}\Omega = 0, \qquad \Omega \wedge \Omega = 0.$$

The precise relationship between these two dual descriptions requires further investigation. In all cases the solutions are encoded in a Riemann–Hilbert problem in the corresponding loop group.

This work raises a number of further question. For example, it should be straightforward to examine the symmetries of these integrable systems using these methods, with the Hamiltonian vector fields playing the role of the symmetry generators (see, for example [T94b]). This would provide a geometrical description of various W_{∞} and W_{KP} algebras. One use of such symmetries is in the construction of a constrained KP-hierarchy. One example of this contains the KdV hierarchy, However, the KdV hierarchy has been shown to be a reduction of the self-dual Yang–Mills equations (and its generalisations). Thus there are two way of looking at the KdV equation: one based on the Yang–Mills self-duality equation and one based on the deformed differential geometry constructed in Section 2. Precisely how these two seemingly different constructions are related deserves further study. In connection with this is how to understand the non-local nature of the Riemann-Hilbert problem for the KP equation compared with the local one for the KdV equation [AC,M].

All the examples of integrable systems in this paper have used Hamiltonian vector fieds in their construction. Are there any systems which use more general, non-Hamiltonian, vector fields? The property of commuting flows for these hierarchies can be traced back to the Jacobi identity for Hamiltonian vector fields, so any hierarchy based on non-Hamiltonian vector fields might lose this property.

One possible use of this deformed calculus would be to develope a theory of deformed (or quantum) twistor spaces (which would encode the Riemann–Hilbert problems in the Moyal loop group) more axiomatically. One obvious place to start is to deform the sympletic structure on the fibres of the non-linear graviton's twistor space. An observation that might be of use is that $\star$-product do exist on the complex manifold $\mathbb{CP}^3$ and other complex coset spaces. This suggests that one should develop a deformation theory (in the sense of Kodaira) of such spaces. Such ideas, however, are outside the scope of this paper.

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Note added in proof

Since this paper was written a number of other papers have appeared. In [PP] and [G-CPP] the Moyal deformation of self-dual gravity has been studied using a chiral model approach and in [S96] it was shown that the Toda lattice is a reduction of this Moyal deformed self-dual gravity, a result analogous to the reduction from the standard, underformed, self-dual gravity equations to the Boyer–Finley equation. Other notable papers are [Ke], [KeS], [DM-H] and [R], which develop various connections between discrete systems, geometry and associative $\star$-products.

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